Polynilpotent Multipliers of Some Nilpotent Products of Cyclic Groups

**Azam Hokmabadi, Behrooz Mashayekhy,*
and Fahimeh Mohammadzadeh**

Abstract

In this article, we present an explicit formula for the cth nilpotent multiplier (the Baer invariant with respect to the variety of nilpotent groups of class at most $c \geq 1$) of the nth nilpotent product of some cyclic groups $G = \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$, (m-copies of $\mathbb{Z}$), where $r_{i+1}|r_i$ for $1 \leq i \leq t-1$ and $c \geq n$ such that $(p, r_1) = 1$ for all primes p less than or equal to n. Also, we compute the polynilpotent multiplier of the group G with respect to the polynilpotent variety $\mathcal{N}_{c_1,c_2,\ldots,c_t}$, where $c_1 \geq n$.

2010 *Mathematics Subject Classification*. 20E10, 20E34, 20F12, 20F18.
Key words and phrases. Polynilpotent multiplier, nilpotent product, cyclic group.

*This research was in part supported by a grant from IPM (No. 85200037).

1 Introduction and Motivation

Let G be any group with a free presentation $G \cong F/R$, where F is a free group. Then the Baer invariant of G with respect to the variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},$$

where V is the set of laws of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \langle v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_k) v(f_1, \ldots, f_i, \ldots, f_k)^{-1} |$$
$$r \in R, f_i \in F, v \in V, 1 \leq i \leq k, k \in \mathbb{N} \rangle.$$

For example, if $\mathcal{V}$ is the variety of abelian groups $\mathcal{A}$, then the Baer invariant of the group G will be $(R \cap F')/[R, F]$, which is isomorphic to $M(G)$, the Schur multiplier of G (see [5]). If $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1, c_2, \ldots, c_t}$, then the Baer invariant of a group G with respect to this variety, which we call a polynilpotent multiplier, is as follows:

$$\mathcal{N}_{c_1,c_2,\ldots,c_t} M(G) = \frac{R \cap \gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F)}{[R, \ _{c_1}F, \ _{c_2}\gamma_{c_1+1}(F), \ldots, \ _{c_t}\gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)]}, \qquad (1)$$

where $\gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\cdots(\gamma_{c_1+1}(F))\cdots))$ is the group which is attained from the iterated terms of the lower central series of F. See [4] for the equality

$$[R\mathcal{N}^*{}_{c_1,c_2,\ldots,c_t}F] = [R, \ _{c_1}F, \ _{c_2}\gamma_{c_1+1}(F), \ldots, \ _{c_t}\gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)].$$

Note that the Baer invariant of G is always abelian and independent of the choice of the free presentation for G with respect to a variety $\mathcal{V}$ (see [5]). In particular, if $t = 1$ and $c_1 = c$, then the Baer invariant of G with respect to the variety $\mathcal{N}_c$ is called the c-nilpotent multiplier and given by

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, \ _cF]}.$$

Determining these Baer invariants of groups is known to be very useful for classification of groups into isologism classes with respect to the chosen varieties (see [5]). In 1979, Moghaddam [8] gave a formula for the c-nilpotent multiplier of a direct product of two groups, where $c+1$ is a prime number or 4. Also, in 1998, Ellis [1] presented the formula for all $c \geq 1$. In 1997, Moghaddam and Mashayekhy [7] presented an explicit formula for the c-nilpotent multiplier of a finite abelian group for every $c \geq 1$.

It is known that the nilpotent product is a generalization of the direct product. In 1992, Gupta and Moghaddam [2] calculated the c-nilpotent multiplier of the nilpotent dihedral group of class $n, G_n = \langle x, y | x^2, y^2, [x,y]^{2^{n-1}} \rangle$. It is routine to verify that $G_n \cong \mathbb{Z}_2 \overset{n}{*} \mathbb{Z}_2$. In 2003, Moghaddam, Mashayekhy, and Kayvanfar [9] extended the previous result and calculated the c-nilpotent multiplier of nth nilpotent products of two cyclic groups for n =2, 3 and 4 under some conditions. Also, the second author [6] gave an implicit formula for the c-nilpotent multiplier of a nilpotent product of cyclic groups.

In this paper, we first obtain an explicit formula for the c-nilpotent multiplier of the nth nilpotent product of some cyclic groups $G = \underbrace{\mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}}_{m-copies} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$, where $r_{i+1} \mid r_i$ for $1 \leq i \leq t-1$ and $c \geq n$ such that $(p, r_1) = 1$ for all primes p less than or equal to n. This result extends the works of Moghaddam and Mashayekhy [7] and Moghaddam, Mashayekhy and Kayvanfar [9]. Second, we present an explicit formula for the polynilpotent multiplier of such a group G with respect to the polynilpotent variety $\mathcal{N}_{c_1,c_2,\ldots,c_t}$, where $c_1 \geq n$.

2 Notation and Preliminaries

Definition 2.1. ([3, §11.1 and §12.3]). The basic commutators on the letters $x_1, x_2, \ldots, x_n, \ldots$ are defined as follows:

(i) The letters $x_1, x_2, \ldots, x_n, \ldots$ are basic commutators of weight one, ordered by setting $x_i < x_j$, if $i < j$.
(ii) Having defined the basic commutators of weight less than n, the basic commutators of weight n are defined as $c_k = [c_i, c_j]$, where

(a) c_i and c_j are basic commutators and $w(c_i) + w(c_j) = n$, where $w(c)$ is the weight of c and

(b) $c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.

(iii) The basic commutators of weight n follow those of weights less than n. The basic commutators of weight n are ordered among themselves lexicographically; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight n, then $[b_1, a_1] < [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

Basic commutators are special cases of outer commutators. Outer commutators on the letters $x_1, x_2, \ldots, x_n, \ldots$ are defined inductively as follows:

The letter x_i is an outer commutator word of weight one. If $u = u(x_1, \ldots, x_s)$ and $v = v(x_{s+1}, \ldots, x_{s+t})$ are outer commutator words of weights s and t, then $w(x_1, \ldots, x_{s+t}) = [u(x_1, \ldots, x_s), v(x_{s+1}, \ldots, x_{s+t})]$ is an outer commutator word of weight $s + t$ and will be written $w = [u, v]$.

Theorem 2.2. ([3, §11.2]). *Let F be the free group on $x_1, x_2, \ldots, x_d$, then for all $1 \leq i \leq n$,*

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)}$$

is the free abelian group, and freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on d letters.

Theorem 2.3. ([3, §11.2]). *The number of basic commutators of weight n on d generators is given by the following formula:*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{\frac{n}{m}},$$

where $\mu(m)$ is the Möbius function, which is defined to be

$$\mu(m) = \begin{cases} 1 & if\ m = 1, \\ 0 & if\ m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & if\ m = p_1 \cdots p_s, \end{cases}$$

where p_i's are distinct prime numbers.

Let $G_i = \langle x_i | x_i^{k_i} \rangle$, for $i \in I$, be the cyclic group of order k_i if $k_i > 1$, and the infinite cyclic group if $k_i = 0$. The nth nilpotent product of the family $\{G_i\}_{i \in I}$ is defined as follows (see [10]):

$$\prod_{i \in I}^{\overset{n}{*}} G_i = \frac{\prod_{i \in I}^{*} G_i}{\gamma_{n+1}(\prod_{i \in I}^{*} G_i)},$$

where $\prod_{i \in I}^{*} G_i$ is the free product of the family $\{G_i\}_{i \in I}$.
Let

$$1 \to R_i = \langle x_i^{k_i} \rangle \to F_i = \langle x_i \rangle \to G_i \to 1$$

be a free presentation for G_i. It is routine to check that a free presentation for the nth nilpotent product of $\prod_{i \in I}^{\overset{n}{*}} G_i$ is as follows (see [9]):

$$1 \to R = S\gamma_{n+1}(F) \to F = \prod_{i \in I}^{*} F_i \to \prod_{i \in I}^{\overset{n}{*}} G_i \to 1,$$

where $S = \langle x_i^{k_i} | i \in I \rangle^F$. Therefore, if $c \geq n$, then the c-nilpotent multiplier of $\prod_{i \in I}^{\overset{n}{*}} G_i$ is

$$\mathcal{N}_c M(\prod_{i \in I}^{\overset{n}{*}} G_i) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]} = \frac{\gamma_{c+1}(F)}{[S,\ _cF]\gamma_{c+n+1}(F)} = \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+n+1}(F)},$$

where $\rho_k(S)$ is defined inductively by $\rho_1(S) = S$ and $\rho_{c+1}(S) = [\rho_c(S), F]$.

Lemma 2.4. *If $1 \leq i < r$ and $(p, r) = 1$ for all primes p less than or equal to i, then r divides $\binom{r}{i}$.*

Proof. Clearly $\binom{r}{i} = r(\frac{(r-1)\cdots(r-i+1)}{1\times 2\times\cdots\times i})$ is an integer. For any prime $p \leq i$, $p|(r-1)\cdots(r-i+1)$, since $p \not| r$. Thus, $1\times 2\times\cdots\times i|(r-1)\cdots(r-i+1)$ and, hence, the result holds. $\square$

The following consequences of the collecting process are vital in the proof of our main result.

Lemma 2.5. ([10]). *Let x, y be any elements of a given group and let $c_1, c_2, \ldots$ be the sequence of basic commutators of weights at least two in x and $[x, y]$, in ascending order. Then*

$$[x^n, y] = [x, y]^n c_1^{f_1(n)} c_2^{f_2(n)} \cdots c_i^{f_i(n)} \cdots , \tag{2}$$

where

$$f_i(n) = a_1\binom{n}{1} + a_2\binom{n}{2} + \cdots + a_{w_i}\binom{n}{w_i}, \tag{3}$$

with $a_i \in \mathbb{Z}$ and w_i being the weight of c_i in x and $[x, y]$. If the group is nilpotent, then the expression in (2) gives an identity, and the sequence of commutators terminates.

Lemma 2.6. ([10]). *Let α be a fixed integer and G a nilpotent group of class at most n. If $b_j \in G$ and $r < n$, then*

$$[b_1, .., b_{i-1}, b_i^{\alpha}, b_{i+1}, \ldots, b_r] = [b_1, \ldots, b_r]^{\alpha} c_1^{f_1(\alpha)} c_2^{f_2(\alpha)} \cdots ,$$

where the c_k's are commutators in $b_1, \ldots, b_r$ of weight strictly greater than r, and every b_j, $1 \leq j \leq r$ appears in each commutator c_k, the c_k's listed in ascending order. The f_i's are of the form (3), with $a_j \in \mathbb{Z}$ and $w_i =$ (the weight of c_i on the b_i) $- (r-1)$.

3 Main Results

Keeping the previous notation, let $k_i = 0$, for $1 \le i \le m$, and $k_{m+j} = r_j > 1$ such that $r_{j+1} | r_j$, for $1 \le j \le t$, then $\prod_{i\in I}^{*^n} G_i = \underbrace{\mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}}_{m-copies} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$. In order to compute the c-nilpotent multiplier of the above group, we need two technical lemmas.

Lemma 3.1. *With the above notation and assumption, if* $(p, r_1) = 1$*, for all primes* p *less than or equal to* $l - i$*, then* $\rho_{c+i}(S)\gamma_{c+l}(F)/\rho_{c+i+1}(S)\gamma_{c+l}(F)$ *is the free abelian group with a basis* $D_{i,1} \cup \cdots \cup D_{i,t}$*, where*

$$D_{i,j} = \{b^{r_j}\rho_{c+i+1}(S)\gamma_{c+l}(F)|\ b \text{ is a basic commutator of weight } c+i \text{ on } x_1, \ldots, x_m, \ldots, x_{m+j} \text{ such that } x_{m+j} \text{ appears in } b\},$$

for $1 \le i \le l - 1$ *and* $1 \le j \le t$*.*

Proof. Using the collecting process (see [3, §11.1]), one can easily check that $\rho_{c+i}(S)\gamma_{c+l}(F)/\rho_{c+i+1}(S)\gamma_{c+l}(F)$ is generated by all $b'\rho_{c+i+1}(S)\gamma_{c+l}(F)$, where b' belongs to the set of basic commutators of weight $c+i, \ldots, c+l-1$ on letters $x_1, \cdots, x_m, x_{m+1}, \ldots, x_{m+t}$ such that one of the $x_{m+1}^{r_1}, \ldots, x_{m+t}^{r_t}$ appears in them. It is easy to check that all the above commutators of weight greater than $c + i$ belong to $\rho_{c+i+1}(S)$. Now, we show that if b' is one of the above commutators of weight $c + i$ such that $x_{m+j}^{r_j}$ appears in it, then

$$b' \equiv b^{r_j} \pmod{\rho_{c+i+1}(S)\gamma_{c+l}(F)}, \tag{4}$$

where b is a basic commutator of weight $c+i$ on $x_1, \ldots, x_m, \ldots, x_{m+t}$ such that x_{m+j} appears in it. (Note that b is actually a basic commutator according to the definition, and b' is the same as b, but the letter x_{m+j} with exponent r_j.) In order to prove the above claim, first we use reverse induction on k

$(i+1 \leq k \leq l-1)$ to show that if u is an outer commutator of weight $c+k$ on $x_1, \ldots, x_m, \ldots, x_{m+t}$ such that x_{m+j} appears in u, then

$$u^{r_j} \in \rho_{c+i+1}(S) \pmod{\gamma_{c+l}(F)}. \tag{5}$$

Let $k = l-1$ and $u = [\ldots, x_{m+j}, \ldots]$, then clearly $u^{r_j} \equiv [\ldots, x_{m+j}^{r_j}, \ldots] \in \rho_{c+i+1}(S) \pmod{\gamma_{c+l}(F)}$.

Now, suppose the above property holds for every $k > k'$. We will show that the property (5) holds for k'. Let $u = [u_1, u_2]$ be an outer commutator of weight $c+k'$ on $x_1, \ldots, x_{m+t}$, where x_{m+j} appears in u_1. Then, by Lemma 2.5, we have

$$u^{r_j} \equiv [u_1^{r_j}, u_2](v_1^{f_1(r_j)} \cdots v_h^{f_h(r_j)})^{-1} \pmod{\gamma_{c+l}(F)},$$

where v_s is a basic commutator of weight w_s in u_1 and $[u_1, u_2]$, $1 \leq s \leq h$. Thus, v_s is an outer commutator of weight greater than $c+k'$ and less than $c+l$ on $x_1, \ldots, x_m, \ldots, x_{m+t}$ such that x_{m+j} appears in it. By the hypothesis, since $r_j | r_1$ we have $(p, r_j) = 1$ for all primes p less than or equal to $l-i$. Also, it is easy to see that $w_s \leq (c+l) - (c+k'-1) = l - k' + 1 \leq l - i$. Therefore, by Lemma 2.4, $r_j | f_s(r_j)$, and so, by induction hypothesis, $v_s^{f_s(r_j)} \in \rho_{c+i+1}(S) \pmod{\gamma_{c+l}(F)}$. Hence, by repeating the above process, if $u = [\ldots, x_{m+j}, \ldots]$, then $u^{r_j} \equiv [\ldots, x_{m+j}^{r_j}, \ldots] v_1'^{f_1'(r_j)} \cdots v_h'^{f_h'(r_j)} \in \rho_{c+i+1}(S) \pmod{\gamma_{c+l}(F)}$. Now using (5), Lemma 2.6, and some commutator manipulations, the congruence (4) holds. Therefore, the set $\bigcup_{j=1}^{t} D_{i,j}$ is a generating set for $\rho_{c+i}(S)\gamma_{c+l}(F)/\rho_{c+i+1}(S)\gamma_{c+l}(F)$. On the other hand, by Theorem 2.2, distinct basic commutators are linearly independent and, hence, distinct powers of these basic commutators are also linearly independent. Therefore, the set $\bigcup_{j=1}^{t} D_{i,j}$ is a basis. $\square$

Lemma 3.2. *With the notation and assumption of the previous lemma, if* $(p, r_1) = 1$ *for all primes* p *less than or equal to* $l-1$*, then*

$$\rho_{c+1}(S)\gamma_{c+l}(F)/\gamma_{c+l}(F)$$

is the free abelian group with a basis $\bigcup_{i=1}^{l-1}(\bigcup_{j=1}^{t} D_{i,j})$.

Proof. Put

$$A_i = \frac{\rho_{c+i}(S)\gamma_{c+l}(F)}{\rho_{c+i+1}(S)\gamma_{c+l}(F)}, B_i = \frac{\rho_{c+1}(S)\gamma_{c+l}(F)}{\rho_{c+i+1}(S)\gamma_{c+l}(F)}.$$

Then, clearly the following exact sequence exists for $1 \leq i \leq l-1$

$$0 \to A_i \to B_i \to B_{i-1} \to 0.$$

By Lemma 3.1, B_1 is a free abelian group, so the following exact sequence:

$$0 \to A_2 \to B_2 \to B_1 \to 0$$

splits and, hence, $B_2 \cong A_2 \oplus B_1$. Also, by Lemma 3.1 every A_i is free abelian, so by induction, every B_i is free abelian and

$$\frac{\rho_{c+1}(S)\gamma_{c+l}(F)}{\gamma_{c+l}(F)} = B_{l-1} \cong A_{l-1} \oplus A_{l-2} \oplus \cdots \oplus A_2 \oplus A_1.$$

Now, using the basis for A_i presented in Lemma 3.1, the result holds. □

Now, we are in a position to state and prove the first main result of the paper.

Theorem 3.3. *Let* $G = \underbrace{\mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}}_{m-copies} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$ *be the nth nilpotent product of some cyclic groups, where* r_{i+1} *divides* r_i *for* $1 \leq i \leq t$. *If* $c \geq n$ *and* $(p, r_1) = 1$ *for all primes* p *less than or equal to* n*, then the c-nilpotent multiplier of* G *is isomorphic to*

$$\mathbb{Z}^{(d_m)} \oplus \mathbb{Z}_{r_1}^{(d_{m+1}-d_m)} \oplus \cdots \oplus \mathbb{Z}_{r_t}^{(d_{m+t}-d_{m+t-1})},$$

where $d_m = \sum_{i=1}^{n} \chi_{c+i}(m)$ *and* $\mathbb{Z}_{r_i}^{(d)}$ *denotes the direct sum of* d *copies of the cyclic group* $\mathbb{Z}_{r_i}$.

Proof. Using the previous notation and assumption, we have

$$\mathcal{N}_c M(G) = \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+n+1}(F)} \cong \frac{\gamma_{c+1}(F)/\gamma_{c+n+1}(F)}{\rho_{c+1}(S)\gamma_{c+n+l}(F)/\gamma_{c+n+1}(F)}.$$

Also, by Theorem 2.2, $\gamma_{c+1}(F)/\gamma_{c+n+1}(F)$ is a free abelian group with the basis consisting of all basic commutators of weight $c+1,\ldots,c+n$ on the letters $x_1,\ldots,x_{m+t}$.

Now, by considering the basis presented for $\rho_{c+1}(S)\gamma_{c+n+l}(F)/\gamma_{c+n+1}(F)$ in Lemma 3.2 and note the points that $D_{i,j}$'s are mutually disjoint and the number of elements of $D_{i,j}$ is equal to $\chi_{c+i}(m+j)-\chi_{c+i}(m+j-1)$, the result holds. $\square$

Now the second main result of the paper, which is in turn an extension of the first one, is as follows:

Theorem 3.4. *Let* $G = \underbrace{\mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}}_{m-copies} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$ *be the nth nilpotent product of some cyclic groups, where* r_{i+1} *divides* r_i*, for* $1 \le i \le t$*. If* $(p, r_1) = 1$ *for all primes* p *less than or equal to* n*, then the polynilpotent multiplier with class row* $c_1, c_2, \ldots, c_s$ *of* G *is*

$$\mathcal{N}_{c_1,c_2,\ldots,c_s} M(G) = \mathbb{Z}^{(d_m)} \oplus \mathbb{Z}_{r_1}^{(d_{m+1}-d_m)} \oplus \cdots \oplus \mathbb{Z}_{r_t}^{(d_{m+t}-d_{m+t-1})},$$

where $d_i = \chi_{c_s+1}(\cdots(\chi_{c_2+1}(\sum_{i=1}^{n}\chi_{c_1+i}(m)))\cdots)$*, for* $c_1 \ge n$ *and* $c_2,\ldots,c_s \ge 1$ *and* $1 \le i \le t$*.*

Proof. Let G be a nilpotent group of class $n \le c_1$ with a free presentation $G = F/R$. Since $\gamma_{c_1+1}(F) \le \gamma_{n+1}(F) \le R$, it gives $\mathcal{N}_{c_1} M(G) = \gamma_{c_1+1}(F)/[R,\ _{c_1}F]$. Now, we can consider $\gamma_{c_1+1}(F)/[R,\ _{c_1}F]$ as a free presentation for $\mathcal{N}_{c_1}M(G)$ and, hence,

$$\mathcal{N}_{c_2} M(\mathcal{N}_{c_1} M(G)) = \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}F]}.$$

Therefore, by (1) we have

$$\mathcal{N}_{c_1,c_2} M(G) = \mathcal{N}_{c_2} M(\mathcal{N}_{c_1} M(G)).$$

By continuing the above process, we can show that

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)=\mathcal{N}_{c_t}M(\cdots\mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(G))\cdots).$$

Using Theorem 3.3, $\mathcal{N}_{c_1}M(G)$ is a finitely generated abelian group of the following form:

$$\mathbb{Z}^{(\sum_{i=1}^{n}\chi_{c_1+i}(m))}\oplus\mathbb{Z}_{r_1}^{(\sum_{i=1}^{n}(\chi_{c_1+i}(m+1)-\chi_{c_1+i}(m))}\oplus$$
$$\cdots\oplus\mathbb{Z}_{r_t}^{(\sum_{i=1}^{n}(\chi_{c_1+i}(m+t)-\chi_{c_1+i}(m+t-1))}.$$

Now applying Theorem 3.3 with $n=1$, the result holds. $\square$

Remark 3.5. Let $G=\underbrace{\mathbb{Z}\overset{n}{*}\cdots\overset{n}{*}\mathbb{Z}}_{m-copies}\overset{n}{*}\mathbb{Z}_{s_1}\overset{n}{*}\cdots\overset{n}{*}\mathbb{Z}_{s_t}$ be the nth nilpotent product of some cyclic groups, where the s_i are arbitrary natural numbers, for $1\le i\le t$. If $c\ge n$ and $(p,s_i)=1$ for all primes p less than or equal to n and $1\le i\le t$, then by a similar proof to Lemmas 3.1 and 3.2 and Theorem 3.3, one can compute the c-nilpotent multiplier of G, but the formula is certainly more complicated than the one in Theorem 3.3. For example, if $G=\mathbb{Z}_{s_1}\overset{n}{*}\mathbb{Z}_{s_2}\overset{n}{*}\mathbb{Z}_{s_3}$, then $\mathcal{N}_cM(G)$ is as follows:

$$\mathbb{Z}_{\alpha}^{(\sum_{i=1}^{n}\chi_{c+i}(2))}\oplus\mathbb{Z}_{\beta}^{(\sum_{i=1}^{n}\chi_{c+i}(2))}\oplus\mathbb{Z}_{\gamma}^{(\sum_{i=1}^{n}\chi_{c+i}(2))}\oplus\mathbb{Z}_{\delta}^{(\sum_{i=1}^{n}\chi_{c+i}(3)-3\sum_{i=1}^{n}\chi_{c+i}(2))},$$

where $\alpha=(s_1,s_2)$, $\beta=(s_2,s_3)$, $\gamma=(s_1,s_3)$, $\delta=(s_1,s_2,s_3)$.

Moreover, using the proof of Theorem 3.4 and the above formula twice, we can compute the polynilpotent multiplier with class row c_1,c_2 of G as follows:

$$\mathcal{N}_{c_1,c_2}M(G)=\mathbb{Z}_{\alpha}^{(e_1)}\oplus\mathbb{Z}_{\beta}^{(e_1)}\oplus\mathbb{Z}_{\gamma}^{(e_1)}\oplus\mathbb{Z}_{\delta}^{(e_2)}\oplus\mathbb{Z}_{(\alpha,\beta)}^{(e_3)}\oplus\mathbb{Z}_{(\alpha,\gamma)}^{(e_3)}\oplus\mathbb{Z}_{(\beta,\gamma)}^{(e_3)}$$
$$\oplus\mathbb{Z}_{(\alpha,\delta)}^{(e_4)}\oplus\mathbb{Z}_{(\beta,\delta)}^{(e_4)}\oplus\mathbb{Z}_{(\gamma,\delta)}^{(e_4)}\oplus\mathbb{Z}_{(\alpha,\beta,\gamma)}^{(e_5)}\oplus\mathbb{Z}_{(\alpha,\beta,\delta)}^{(e_6)}\oplus\mathbb{Z}_{(\beta,\gamma,\delta)}^{(e_6)},$$

where

$$e_1=\chi_{c_2+1}(\sum_{i=1}^{n}\chi_{c_1+i}(2)),\ e_2=\chi_{c_2+1}(\sum_{i=1}^{n}\chi_{c+i}(3)-3\sum_{i=1}^{n}\chi_{c+i}(2)),$$

$$e_3=\chi_{c_2+1}(2\sum_{i=1}^{n}\chi_{c_1+i}(2)){-}2e_1,\ e_4=\chi_{c_2+1}(\sum_{i=1}^{n}\chi_{c+i}(3){-}2\sum_{i=1}^{n}\chi_{c+i}(2)){-}e_1{-}e_2,$$

$$e_5=\chi_{c_2+1}(3\sum_{i=1}^{n}\chi_{c_1+i}(2))-3\chi_{c_2+1}(2\sum_{i=1}^{n}\chi_{c_1+i}(2)),$$

$$e_6=\chi_{c_2+1}(\sum_{i=1}^{n}\chi_{c+i}(3)-\sum_{i=1}^{n}\chi_{c+i}(2))-\chi_{c_2+1}(2\sum_{i=1}^{n}\chi_{c_1+i}(2))-$$

$$\chi_{c_2+1}(\sum_{i=1}^{n}\chi_{c+i}(3)-2\sum_{i=1}^{n}\chi_{c+i}(2)).$$

It seems that the general formula in this case is more complicated than to write!

References

[1] G. Ellis, *On groups with a finite nilpotent upper central quotient*, Arch. Math. 70 (1998), 89–96.

[2] N. D. Gupta and M. R. R. Moghaddam, *Higher Schur multiplicators of nilpotent dihedral groups*, C. R. Math. Rep. Acad. Sci. Canada XIV 5 (1992), 225–230.

[3] M. Hall, The theory of groups. Macmillan Company, New York, 1959.

[4] N. S. Hekster, *Varieties of groups and isologisms*, J. Austral. Math. Soc. (Ser. A) 46 (1989), 22–60.

[5] C. R. Leedham-Green and S. Mckay, *Baer-invariants, isologism, varietal laws and homology*, Acta Math. 137 (1976), 99–150.

[6] B. Mashayekhy, *Some notes on the Baer-invariant of a nilpotent product of groups*, J. Algebra 235 (2001), 15–26.

[7] M. R. R. Mogaddam and B. Mashayekhy, *Higher Schur multiplicator of a finite abelian group*, Algebra Colloq. 4 (3) (1997), 317–322.

[8] M. R. R. Moghaddam, *The Baer-invariant of a direct product*, Arch. Math. 33 (1979), 504–511.

[9] M. R. R. Moghaddam, B. Mashayekhy and S. Kayvanfar, *The higher Schur multiplicator of certain class of groups*, Southeast Asian Bull. Math. 27 (2003), 121–128.

[10] R. R. Struik, *On nilpotent products of cyclic groups*, Canad. J. Math. 12 (1960), 447–462.

Department of Pure Mathematics, P. O. Box 1159-91775, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail address: hokmabadi_ah@yahoo.com

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Iran, and Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran.
E-mail address: bmashf@um.ac.ir

Department of Pure Mathematics, P. O. Box 1159-91775, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail address: fa36407@yahoo.com